Exponential stability and periodic solutions of neural networks with continuously distributed delays

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In this paper we study a class of neural networks with continuously distributed delays. By means the of Lyapunov functional method, we obtain some sufficient conditions ensuring the existence, uniqueness, and global exponential stability of the equilibrium and periodic solution. We also estimate the exponentially convergent rate. Our results are less restrictive than previously known criteria and can be applied to neural networks with a broad range of activation functions assuming neither differentiability nor strict monotonicity. Moreover, these conclusions are presented in terms of system parameters and can be easily verified. Therefore, our results play an important role in the design of globally exponentially stable neural circuits and periodic oscillatory neural circuits.

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I. INTRODUCTION

It is well known that for neural networks with delays, it is rather difficult to analyze their stability properties due to introduction of delays. There are usually two ways to do this. One is to linearize the system near equilibrium (the original system has the same stability properties as the linearized system near equilibrium); conditions obtained in this way concern the local stability around an equilibrium. Another way is to construct a suitable Lyapunov function for the system and then to derive sufficient conditions ensuring stability, this usually involves global stability. Of course, constructing a suitable Lyapunov function is usually not an easy task. Marcus and Westervelt [1] studied the stability of analog neural networks with delay by linearizing the systems. By using the Lyapunov functional approach, Gopalsamy and He [2] and Lu [3] established some sufficient conditions for the delay-independent stability; Cao and co-worker [4,5] researched the global exponential stability. In addition, in the applications of neural networks, it is required that the convergent rate should be improved in order to reduce the span that neurons need to calculate.

Although the use of constant discrete delays in models with delayed feedbacks provides a good approximation to simple circuits consisting of a small number of neurons, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Thus, there will be a distribution of propagation delays. In this case, the signal propagation is no longer instantaneous and cannot be modeled with discrete time delay. A more appropriate way is to incorporate distributed delays. Tank and Hopfield [6] have proposed a neural circuit with distributed delays, which solves a general problem of recognizing patterns in a time-dependent signal. For the applications of neural networks with distributed delays as described delays as described in Ref. [6], the readers may also refer to Refs. [7,8]. Moreover, a neural network model with distributed delay is more general than that with discrete delay. This is because the distributed delay becomes a discrete delay when the delay kernel is a δ function at a certain time.

In this paper, we investigate a class of neural networks with continuously distributed delays modeled by the following system of delayed differential equations:

$$x_i'(t) = -\mu_i x_i(t) + \sum_{j=1}^{n}\left[a_{ij} f_j(x_j(t)) + b_{ij} \times g_j\left(\int_0^\infty k_j(s) x_j(t-s)\,ds\right)\right] + I_i(t), t \geqslant t_0, \quad i = 1,2,\ldots,n, \tag{1}$$

where n corresponds to the number of neurons in the neural network, $x_i(t)$ and $I_i(t)$ represent the activation and external inputs of the ith neuron, respectively, a_{ij} and b_{ij} represent the strengths of synaptical connections and are constants, $\mu_i > 0$ represents the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs, k_j denotes the refractoriness of the jth neuron after it has fired or responded, $f_j, g_j (j = 1,2,\ldots,n)$ are signal transmission functions of the jth neuron.

In the following we assume that $k_j, I_i : [t_0,\infty) \to R$ are continuous, k_j is integrable and satisfies

$$\int_0^\infty k_j(s)\,ds = c_j > 0, \quad \int_0^\infty s k_j(s)\,ds < \infty, \quad j = 1,2,\ldots,n,$$

the signal transmission functions $f_j, g_j (j = 1,2,\ldots,n)$ possess the following properties:

(H1) $f_j, g_j (j = 1,2,\cdots,n)$ are bounded on R;

(H2) there exists $p_j > 0$ and $q_j > 0$ such that $|f_j(u) - f_j(v)| \leqslant p_i |u - v|$ and $|g_i(x) - g_i(y)| \leqslant q_i |x - y|$ for any $u,v \in R$ and $j = 1,2,\ldots,n$.

The initial conditions associated with system (1) are of the form

$$x_i(s) = \varphi_i(s), \quad s \in (-\infty, 0], \tag{2}$$

where $\varphi_i : (-\infty, 0] \to R$ is usually assumed to be continuous.

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Our results impose milder constraints on system (1) and apply to neural networks with a broad range of signal transmission functions assuming neither differentiability nor strict monotonicity.

II. STABILITY ANALYSIS

Consider the special case of model (1) as $I_i(t) \equiv I_i$, i.e.,

$$x_i'(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \left[a_{ij} f_j(x_j(t)) + b_{ij} \times g_j\left(\int_0^\infty k_j(s) x_j(t-s) ds \right) \right] + I_i, \quad (3)$$

where $t \geqslant t_0$, $i=1,2,\ldots,n$, and I_i is constant.

Lemma 1. Assume that the signal transmission functions $f_j, g_j (j=1,2,\ldots,n)$ satisfy (H1) and (H2). Then there exists an equilibrium for model (3).

Proof. Clearly, $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is an equilibrium of Eq. (3) if and only if x^* satisfies that

$$\mu_i x_i^* = \sum_{j=1}^{n} \left[a_{ij} f_j(x_j^*) + b_{ij} g_j\left(\int_0^\infty k_j(s) x_j^* ds \right) \right] + I_i = \sum_{j=1}^{n} [a_{ij} f_j(x_j^*) + b_{ij} g_j(c_j x_j^*)] + I_i.$$

Construct a mapping to be

$$F(u_1, u_2, \ldots, u_n) = (F_1(u_1), F_2(u_2), \ldots, F_n(u_n))^T,$$

where $F_i(u_i) = \mu_i^{-1}(\Sigma_{j=1}^n [a_{ij} f_j(u_j) + b_{ij} g_j(c_j u_j)] + I_i), i = 1,2,\ldots,n$. Then it is easy to see that x^* is the equilibrium of system (3) if and only if x^* is the fixed point of $F(u)$. Consider the following closed convex set:

$$\Omega = \left\{ x \in R^n; |x_i - \mu_i^{-1} I_i| \leqslant \mu_i^{-1} \sum_{j=1}^{n} [a_{ij} f_j^+ + b_{ij} g_j^+], \quad i=1,2,\ldots,n \right\},$$

where $f_j^+ := \sup_{s \in R}\{|f_j(s)|\}$ and $g_j^+ := \sup_{s \in R}\{|g_j(s)|\}$. It is easy to verify that F is continuous and satisfies that $F(u) \in \Omega$ for all $u \in \Omega$. Hence by the Brouwer's theorem, F has at least one fixed point x^*. This completes the proof.

The above lemma indicates the existence of the equilibrium x^* of system (3), but is not sufficient to guarantee its uniqueness. Our purpose in this section is to obtain some sufficient conditions ensuring the uniqueness and even the global exponential stability of equilibrium x^*. It is an easy exercise to verify the following lemma.

Lemma 2. $\alpha y x^{\alpha-1} \leqslant y^\alpha + (\alpha-1) x^\alpha$ for all $\alpha > 1, x, y \geqslant 0$.

Thus, we state one of main results as follows:

Theorem 1. Assume that the signal transmission functions $f_j, g_j (j=1,2,\ldots,n)$ satisfy (H1) and (H2). If there exist positive constants $d_1, d_2, \ldots, d_n$ and $\alpha > 1$ such that one of the following conditions (A1)–(A6) holds, then the equilibrium x^* of system (3) is globally exponentially stable and hence unique.

$$(\text{A1}) \max_{1 \leqslant i \leqslant n} \left\{ \frac{1}{\alpha \mu_i} \sum_{j=1}^{n} \left[(\alpha-1)(|a_{ij}| p_j + |b_{ij}| q_j c_j) + \frac{1}{d_i}(|a_{ji}| d_j p_i + |b_{ji}| d_j q_i c_i) \right] \right\} < 1;$$

$$(\text{A2}) \max_{1 \leqslant i \leqslant n} \left\{ \frac{1}{\alpha \mu_i} \sum_{j=1}^{n} \left[(\alpha-1)(|a_{ij}| p_j + c_j) + \frac{1}{d_i}(|a_{ji}| d_j p_i + |b_{ji}|^\alpha d_j q_i^\alpha c_i) \right] \right\} < 1;$$

$$(\text{A3}) \max_{1 \leqslant i \leqslant n} \left\{ \frac{1}{\alpha \mu_i} \sum_{j=1}^{n} \left[(\alpha-1)(|a_{ij}| p_j + |b_{ij}| c_j) + \frac{1}{d_i}(|a_{ji}| d_j p_i + |b_{ji}| d_j q_i^\alpha c_i) \right] \right\} < 1;$$

$$(\text{A4}) \max_{1 \leqslant i \leqslant n} \left\{ \frac{1}{\alpha \mu_i} \sum_{j=1}^{n} \left[(\alpha-1)(|a_{ij}| p_j + q_j c_j) + \frac{1}{d_i}(|a_{ji}| d_j p_i + |b_{ji}|^\alpha d_j q_i c_i) \right] \right\} < 1;$$

$$(\text{A5}) \max_{1 \leqslant i \leqslant n} \left\{ \frac{1}{\alpha \mu_i} \sum_{j=1}^{n} \left[(\alpha-1)(|a_{ij}| p_j + |b_{ij}| q_j^{\alpha/(\alpha-1)} c_j) + \frac{1}{d_i}(|a_{ji}| d_j p_i + |b_{ji}| d_j c_i) \right] \right\} < 1;$$

$$(\text{A6}) \max_{1 \leqslant i \leqslant n} \left\{ \frac{1}{\alpha \mu_i} \sum_{j=1}^{n} \left[(\alpha-1)(|a_{ij}| p_j + q_j^{\alpha/(\alpha-1)} c_j) + \frac{1}{d_i}(|a_{ji}| d_j p_i + |b_{ji}|^\alpha d_j c_i) \right] \right\} < 1.$$

Proof. Since x^* is an equilibrium of system (3), then the deviations $y_i(t) = x_i(t) - x^*$, $i=1,2,\ldots,n$ satisfy

$$y_i'(t) = -\mu_i y_i(t) + \sum_{j=1}^{n} a_{ij}[f_j(x_j^* + y_j(t)) - f_j(x_j^*)] + \sum_{j=1}^{n} b_{ij}\left[g_j\left(\int_0^\infty k_j(s)(x_j^* + y_j(t-s)) ds \right) - g_j(c_j x_j^*) \right] + I_i. \quad (4)$$

Obviously, system (4) has an equilibrium 0. Hence, to prove the global exponential stability of model (3), it is equivalent to prove the global exponential stability of Eq.

(4). In what follows, we distinguish three cases to prove that each of the conditions (A1)–(A6) ensures the global exponential stability of Eq. (4).

Case 1. Assume that condition (A1) holds. Then there exists a positive constant $\lambda<\mu_i$ such that

$$\mu_i-\lambda>(\alpha-1)\alpha^{-1}\sum_{j=1}^{n}\left[|a_{ij}|p_j+|b_{ij}|q_j\int_0^\infty k_j(s)e^{\lambda s}ds\right]+(\alpha d_i)^{-1}\sum_{j=1}^{n}\left[|a_{ji}|d_jp_i+|b_{ji}|d_jq_i\int_0^\infty k_i(s)e^{\lambda s}ds\right].$$

For all $i=1,2,\ldots,n$, substituting $z_i(t)=y_i(t)e^{\lambda t}$ if $t\geqslant 0$ and $z_i(t)=y_i(t)$ otherwise into Eq. (4), we have

$$z_i'(t)=(\lambda-\mu_i)z_i(t)+e^{\lambda t}\sum_{j=1}^{n}a_{ij}[f_j(x_j^*+z_j(t)e^{-\lambda t})-f_j(x_j^*)]+e^{\lambda t}\sum_{j=1}^{n}b_{ij}\left[g_j\left(\int_0^\infty k_j(s)[x_j^*+z_j(t-s)e^{\lambda(s-t)}]ds\right)-g_j(c_jx_j^*)\right]+I_i. \tag{5}$$

We consider a Lyapunov functional $V_1(t)=V_1(z)(t)$ defined by

$$V_1(t)=\alpha^{-1}\sum_{i=1}^{n}d_i\left[|z_i(t)|^\alpha+\sum_{j=1}^{n}|b_{ij}|q_j\times\int_0^\infty\int_{t-s}^{t}k_i(s)e^{\lambda s}|z_j(\xi)|^\alpha d\xi ds\right].$$

Calculating the upper right derivative D^+V_1 of V_1 along the solutions of system (5), we have

$$\begin{aligned}
D^+V_1(t)&=\sum_{i=1}^{n}d_i|z_i(t)|^{\alpha-1}D^+|z_i(t)|+\alpha^{-1}\sum_{i=1}^{n}d_i\sum_{j=1}^{n}|b_{ij}|q_j\int_0^\infty k_i(s)e^{\lambda s}[|z_j(t)|^\alpha-|z_j(t-s)|^\alpha]ds\\
&\leqslant\sum_{i=1}^{n}d_i\Bigg[(\lambda-\mu_i)|z_i(t)|^\alpha+\sum_{j=1}^{n}|a_{ij}|p_j|z_i(t)|^{\alpha-1}|z_j(t)|+\sum_{j=1}^{n}|b_{ij}|q_j|z_i(t)|^{\alpha-1}\int_0^\infty k_i(s)|z_j(t-s)|\\
&\qquad\times e^{\lambda s}ds+\alpha^{-1}\sum_{j=1}^{n}|b_{ij}|q_j\int_0^\infty k_i(s)e^{\lambda s}[|z_j(t)|^\alpha-|z_j(t-s)|^\alpha]ds\Bigg]\\
&\leqslant\sum_{i=1}^{n}d_i\Bigg[(\lambda-\mu_i)|z_i(t)|^\alpha+\alpha^{-1}\sum_{j=1}^{n}|a_{ij}|p_j[(\alpha-1)|z_i(t)|^\alpha+|z_j(t)|^\alpha]+\alpha^{-1}\sum_{j=1}^{n}|b_{ij}|q_j\\
&\qquad\times\int_0^\infty k_i(s)[(\alpha-1)|z_i(t)|^\alpha+|z_j(t-s)|^\alpha]e^{\lambda s}ds+\alpha^{-1}\sum_{j=1}^{n}|b_{ij}|q_j\int_0^\infty k_i(s)e^{\lambda s}[|z_j(t)|^\alpha-|z_j(t-s)|^\alpha]ds\Bigg]\\
&\leqslant\sum_{i=1}^{n}d_i\Bigg\{\lambda-\mu_i+\alpha^{-1}(\alpha-1)\sum_{j=1}^{n}\left[|a_{ij}|p_j+|b_{ij}|q_j\int_0^\infty k_j(s)e^{\lambda s}ds\right]+(\alpha d_i)^{-1}\sum_{j=1}^{n}\Bigg[|a_{ji}|p_id_j+|b_{ji}|q_id_j\\
&\qquad\times\int_0^\infty k_i(s)e^{\lambda s}ds\Bigg]\Bigg\}|z_i(t)|^\alpha\leqslant 0.
\end{aligned}$$

That is, we have

$$V_1(t)\leqslant V_1(0).$$

Therefore, we obtain

$$\begin{aligned}
\sum_{i=1}^{n}d_i|z_i(t)|^\alpha&\leqslant\sum_{i=1}^{n}d_i\left[|z_i(0)|^\alpha+\sum_{j=1}^{n}|b_{ij}|q_j\int_0^\infty\int_{-s}^{0}k_i(s)e^{\lambda s}|z_j(\xi)|^\alpha d\xi ds\right]\\
&\leqslant\sum_{i=1}^{n}d_i\left[1+\sum_{j=1}^{n}|b_{ij}|q_j\int_0^\infty k_i(s)se^{\lambda s}ds\right]\sup_{\xi\leqslant 0}\{|z_i(\xi)|^\alpha\}.
\end{aligned}$$

Thus,

$$|y_i(t)|\leqslant\left\{d_i^{-1}\sum_{i=1}^n d_i\left[1+\sum_{j=1}^n |b_{ij}|q_j\int_0^\infty k_i(s)se^{\lambda s}ds\right]\right\}^{1/\alpha}\sup_{\xi\leqslant 0}\{|\varphi_i(\xi)-x^*|\}e^{-\lambda t}$$

for all $t\geqslant 0$ and $i=1,2,\ldots,n$. This implies that the equilibrium x^* is globally exponentially stable. Moreover, the exponential convergent rate is λ.

Case 2. Assume that condition (A2) holds. Then there exists a positive constant $\lambda<\mu_i$ such that

$$\mu_i-\lambda>(\alpha-1)\alpha^{-1}\sum_{j=1}^n\left(|a_{ij}|p_j+\int_0^\infty k_j(s)e^{\lambda s}ds\right)+(\alpha d_i)^{-1}\sum_{j=1}^n\left[|a_{ji}|d_jp_i+|b_{ji}|^\alpha d_jq_i^{\alpha/(\alpha-1)}\times\int_0^\infty k_i(s)e^{\lambda s}ds\right].$$

Consider the transformed system (5) and a Lyapunov functional $V_2(t)=V_2(z)(t)$ defined by

$$V_2(t)=\alpha^{-1}\sum_{i=1}^n d_i\left[|z_i(t)|^\alpha+\sum_{j=1}^n |b_{ij}|^\alpha q_j^\alpha\times\int_0^\infty\int_{t-s}^t k_i(s)e^{\lambda s}|z_j(\xi)|^\alpha d\xi ds\right].$$

We calculate the upper right derivative D^+V_2 of V_2 along the solutions of system (5). Using arguments similar to case 1, we have

$$|y_i(t)|\leqslant\left\{d_i^{-1}\sum_{i=1}^n d_i\left[1+\sum_{j=1}^n |b_{ij}|^\alpha q_j^\alpha\times\int_0^\infty k_i(s)se^{\lambda s}ds\right]\right\}^{1/\alpha}\sup_{\xi\leqslant 0}\{|\varphi_i(\xi)-x^*|\}e^{-\lambda t}$$

for all $t\geqslant 0$ and $i=1,2,\ldots,n$. This implies that the equilibrium x^* is globally exponentially stable. Moreover, the exponential convergent rate is λ.

Case 3. Assume that one of conditions (A3)–(A6) holds. Consider the associated Lyapunov functions defined by

$$V_3(t)=\alpha^{-1}\sum_{i=1}^n d_i\left[|z_i(t)|^\alpha+\sum_{j=1}^n |b_{ij}|q_j^\alpha\times\int_0^\infty\int_{t-s}^t k_i(s)e^{\lambda s}|z_j(\xi)|^\alpha d\xi ds\right],$$

$$V_4(t)=\alpha^{-1}\sum_{i=1}^n d_i\left[|z_i(t)|^\alpha+\sum_{j=1}^n |b_{ij}|^\alpha q_j\times\int_0^\infty\int_{t-s}^t k_i(s)e^{\lambda s}|z_j(\xi)|^\alpha d\xi ds\right],$$

$$V_5(t)=\alpha^{-1}\sum_{i=1}^n d_i\left[|z_i(t)|^\alpha+\sum_{j=1}^n |b_{ij}|\times\int_0^\infty\int_{t-s}^t k_i(s)e^{\lambda s}|z_j(\xi)|^\alpha d\xi ds\right],$$

$$V_6(t)=\alpha^{-1}\sum_{i=1}^n d_i\left[|z_i(t)|^\alpha+\sum_{j=1}^n |b_{ij}|^\alpha\times\int_0^\infty\int_{t-s}^t k_i(s)e^{\lambda s}|z_j(\xi)|^\alpha d\xi ds\right].$$

Using arguments similar to case 1, we, respectively, have

$$|y_i(t)|\leqslant\left\{d_i^{-1}\sum_{i=1}^n d_i\left[1+\sum_{j=1}^n |b_{ij}|q_j^\alpha\int_0^\infty k_i(s)se^{\lambda s}ds\right]\right\}^{1/\alpha}\times\sup_{\xi\leqslant 0}\{|\varphi_i(\xi)-x^*|\}e^{-\lambda t},$$

$$|y_i(t)|\leqslant\left\{d_i^{-1}\sum_{i=1}^n d_i\left[1+\sum_{j=1}^n |b_{ij}|^\alpha q_j\int_0^\infty k_i(s)se^{\lambda s}ds\right]\right\}^{1/\alpha}\times\sup_{\xi\leqslant 0}\{|\varphi_i(\xi)-x^*|\}e^{-\lambda t},$$

$$|y_i(t)|\leqslant\left\{d_i^{-1}\sum_{i=1}^n d_i\left[1+\sum_{j=1}^n |b_{ij}|\int_0^\infty k_i(s)se^{\lambda s}ds\right]\right\}^{1/\alpha}\times\sup_{\xi\leqslant 0}\{|\varphi_i(\xi)-x^*|\}e^{-\lambda t},$$

$$|y_i(t)|\leqslant\left\{d_i^{-1}\sum_{i=1}^n d_i\left[1+\sum_{j=1}^n |b_{ij}|^\alpha\int_0^\infty k_i(s)se^{\lambda s}ds\right]\right\}^{1/\alpha}\times\sup_{\xi\leqslant 0}\{|\varphi_i(\xi)-x^*|\}e^{-\lambda t}$$

for all $t\geqslant 0$ and $i=1,2,\ldots,n$. This implies that the equilibrium x^* is globally exponentially stable.

Corollary 1. Assume that (H1) and (H2) hold. If there exist positive constants $d_1, d_2, \ldots, d_n$ such that one of the following conditions (i)–(viii) holds, then system (3) is globally exponentially stable:

$$\text{(i)}\quad 2\mu_i > \sum_{j=1}^{n} (|a_{ij}|p_j + |b_{ij}|q_j c_j + d_i^{-1}|a_{ji}|d_j p_i + d_i^{-1}|b_{ji}|d_j q_i c_i) \quad \text{for all } i=1,2,\ldots,n;$$

$$\text{(ii)}\quad \mu_i > d_i^{-1}\sum_{j=1}^{n} (|a_{ji}|d_j p_i + |b_{ji}|d_j q_i c_i) \quad \text{for all } i=1,2,\ldots,n;$$

$$\text{(iii)}\quad \mu_i > \sum_{j=1}^{n} (|a_{ij}|p_j + |b_{ij}|q_j c_j) \quad \text{for all } i=1,2,\ldots,n;$$

$$\text{(iv)}\quad 2\mu_i > \sum_{j=1}^{n} (|a_{ij}|p_j + c_j + d_i^{-1}|a_{ji}|d_j p_i + d_i^{-1}|b_{ji}|^2 d_j q_i^2 c_i) \quad \text{for all } i=1,2,\ldots,n;$$

$$\text{(v)}\quad 2\mu_i > \sum_{j=1}^{n} (|a_{ij}|p_j + |b_{ij}|c_j + d_i^{-1}|a_{ji}|d_j p_i + d_i^{-1}|b_{ji}|d_j q_i^2 c_i) \quad \text{for all } i=1,2,\ldots,n;$$

$$\text{(vi)}\quad 2\mu_i > \sum_{j=1}^{n} (|a_{ij}|p_j + q_j c_j + d_i^{-1}|a_{ji}|d_j p_i + d_i^{-1}|b_{ji}|^2 d_j q_i c_i) \quad \text{for all } i=1,2,\ldots,n;$$

$$\text{(vii)}\quad 2\mu_i > \sum_{j=1}^{n} (|a_{ij}|p_j + |b_{ij}|q_j^2 c_j + d_i^{-1}|a_{ji}|d_j p_i + d_i^{-1}|b_{ji}|d_j c_i) \quad \text{for all } i=1,2,\ldots,n;$$

$$\text{(viii)}\quad 2\mu_i > \sum_{j=1}^{n} (|a_{ij}|p_j + q_j^2 c_j + d_i^{-1}|a_{ji}|d_j p_i + d_i^{-1}|b_{ji}|^2 d_j c_i) \quad \text{for all } i=1,2,\ldots,n.$$

Proof. First, condition (i) is a special case of (A1) as $\alpha = 2$. Next, if condition (ii) or (iii) holds, then there exists a number $\alpha > 1$ such that (A1) holds. Finally, the remaining conditions can be verified similarly.

III. EXISTENCE OF PERIODIC SOLUTIONS

In this section we investigate the periodic solutions of the model of the form

$$x_i'(t) = -\mu_i x_i(t) + \sum_{j=1}^{n}\left[a_{ij} f_j(x_j(t)) + b_{ij} \times g_j\left(\int_0^\infty k_j(s) x_j(t-s)ds\right)\right] + I_i(t), \tag{6}$$

where $t \geqslant t_0$, $i=1,2,\ldots,n$ and $I_i: R^+ \to R$ is a continuously periodic function with period ω, i.e., $I_i(t+\omega) = I_i(t)$.

Theorem 2. Assume that (H1) and (H2) hold. If there exist positive constants $d_1, d_2, \ldots, d_n$ and $\alpha > 1$ such that one of conditions (A1)–(A6) holds, then there exists exactly one ω-periodic solution of system (6) and all other solutions of (6) converge exponentially to it as $t \to \infty$.

Proof. Let $C = C((-\infty,0],R^n)$ be the Banach space of continuous functions that map $(-\infty,0]$ into R^n with the topology of uniform convergence. For any $\varphi \in C$, we define $\|\varphi\| = \sup_{t \leqslant 0}|\varphi(\theta)|$, where $|\varphi(\theta)| = \max_{1\leqslant i \leqslant n}|\varphi_i(\theta)|$.

For any $\varphi, \psi \in C$, we denote the solutions of Eq. (6) with the initial values $(0,\varphi)$ and $(0,\psi)$ as

$$x(t,\varphi) = (x_1(t,\varphi), x_2(t,\varphi), \ldots, x_n(t,\varphi))^T,$$

$$x(t,\psi) = (x_1(t,\psi), x_2(t,\psi), \cdots, x_n(t,\psi))^T,$$

respectively. For a given $s \in [0,\infty)$ and a continuous function $x:[-\infty,\infty) \to R^n$, we define $x_s:[-\infty,0] \to R^n$ by $x_s(\theta) = x(s+\theta)$ for $\theta \in [-\infty,0]$.

Thus, it follows from Eq. (6) that

$$[x_i(t,\varphi) - x_i(t,\psi)]' = -\mu_i[x_i(t,\varphi) - x_i(t,\psi)] + \sum_{j=1}^{n} a_{ij}[f_j(x_j(t,\varphi)) - f_j(x_j(t,\psi))] + \sum_{j=1}^{n} b_{ij}\left[g_j\left(\int_0^\infty k_j(s)x_j(t-s,\varphi)ds\right) - g_j\left(\int_0^\infty k_j(s)x_j(t-s,\varphi)ds\right)\right]$$

for all $t \geqslant t_0$, $i=1,2,\ldots,n$.

Define

$$y_i(t) = x_i(t,\varphi) - x_i(t,\psi), \quad i=1,2,\ldots,n.$$

One can derive that

$$y_i'(t) = -\mu_i y_i(t) + \sum_{j=1}^{n}\left[a_{ij} f_j'(\theta_j(t)) y_i(t) + b_{ij} g_j'(\delta_j(t)) \int_0^\infty k_j(s) y_j(t-s)ds\right] \tag{7}$$

for all $t \geqslant t_0$, $i=1,2,\ldots,n$, where $\theta_j(t)$ lies between $x_j(t,\varphi)$ and $x_j(t,\psi)$, $\delta_j(t)$ lies between $\int_0^\infty k_j(s)x_j(t-s,\varphi)ds$ and $\int_0^\infty k_j(s)x_j(t-s,\psi)ds$. Using similar arguments as that in the proof of Theorem 1, we can deduce that 0 is the global exponential stable equilibrium of system (7), i.e.,

$$|y_i(t)| \leqslant K e^{-\lambda t}\|\varphi - \psi\|, \quad i=1,2,\ldots,n,$$

where $K > 1$ is a constant and $\lambda > 0$ is determined by condition (H3). Therefore,

$$|x_i(t,\varphi) - x_i(t,\psi)| \leqslant K e^{-\lambda t}\|\varphi - \psi\|, \quad i=1,2,\ldots,n.$$

Finally, the rest of the proof is similar to that of Theorem 4 in Ref. [9] and is hence omitted here.

Corollary 2. Under the assumptions of Corollary 1, there exists exactly one ω-periodic solution of system (6) and all other solutions of Eq. (6) converge exponentially to it as $t \to \infty$.

IV. EXAMPLES

Example 1. Consider the following neural networks:

$$x_1'(t) = -0.6x_1(t) + 0.1f(x_1(t)) + 0.1f(x_2(t)) + 0.1f\left(\int_0^\infty e^{-s}x_1(t-s)ds\right) + 0.2f\left(\int_0^\infty e^{-s}x_2(t-s)ds\right) - 1.1, \tag{8}$$

$$x_2'(t) = -0.7x_2(t) + 0.1f(x_1(t)) + 0.1f(x_2(t)) + 0.2f\left(\int_0^\infty e^{-s}x_1(t-s)ds\right) + 0.2f\left(\int_0^\infty e^{-s}x_2(t-s)ds\right) - 1.4,$$

where the signal transmission function is described by a piecewise-linear function $f(x) = \frac{1}{2}(|x+1| - |x-1|)$, the continuous delay kernel $k_j(s) = e^{-s}$. It is easy to see that f clearly satisfies the hypotheses (H1) and (H2) above, $p_i = q_i = 1 (i = 1,2)$, and $k_j(s)$ satisfies that $\int_0^\infty e^{-s}ds = 1 > 0$ and $\int_0^\infty se^{-s}ds = 1 < \infty$. Hence, the condition (A1) in Theorem 1 can be reduced to

$$(0.1\alpha + 0.3)d_1 > 0.4d_2, \; 0.4d_2 > 0.3d_1 .$$

In fact, we can take $\alpha = 2, d_1 = d_2 = 1$. Thus by Theorem 1 the equilibrium of model (8) is globally exponentially stable. It is easy to verify that (1,1) is an equilibrium of the model, and the exponential convergent rate is greater than 0.0353.

Example 2. Consider the following neural networks with delays.

$$x_1'(t) = -x_1(t) + 0.1f(x_1(t)) + 0.2f(x_2(t)) + 0.2f\left(\int_0^\infty e^{-s}x_1(t-s)ds\right) + 0.4f\left(\int_0^\infty e^{-s}x_2(t-s)ds\right) + I_1(t), \tag{9}$$

$$x_2'(t) = -x_2(t) + 0.2f(x_1(t)) + 0.1f(x_2(t)) + 0.3f\left(\int_0^\infty e^{-s}x_1(t-s)ds\right) + 0.2f\left(\int_0^\infty e^{-s}x_2(t-s)ds\right) + I_2(t),$$

where $f(x) = \tanh(x)$,

$$I_1(t) = \cos t + \sin t - 0.1\tanh(\sin t) - 0.2\tanh(\cos t) - 0.2\tanh\left(\frac{\sin t - \cos t}{2}\right) - 0.4\tanh\left(\frac{\sin t + \cos t}{2}\right),$$

$$I_2(t) = \cos t - \sin t - 0.2\tanh(\sin t) - 0.1\tanh(\cos t) - 0.3\tanh\left(\frac{\sin t - \cos t}{2}\right) - 0.2\tanh\left(\frac{\sin t + \cos t}{2}\right).$$

Then, condition (A1) in Theorem 2 can be reduced to

$$(0.1\alpha + 0.6)d_1 > 0.5d_2, \; (0.2\alpha + 0.5)d_2 > 0.6d_1 .$$

In fact, we can take $\alpha = 2, d_1 = 2, d_2 = 3$. Thus by Theorem 2 there exists exactly one 2π-periodic solution of system (9) and all other solutions of (9) converge exponentially to it as $t \to \infty$. It is easy to verify that $(\sin t, \cos t)$ is the 2π-periodic solution of model (9). Moreover, the exponential convergent rate is greater than 0.0076.

V. CONCLUSION

Some sufficient conditions for global exponential stability for a kind of neural networks with continuously distributed delays have been obtained. Since the conditions of Theorems 1 and 2 include some adjustable parameters, the results have a wider adaptive range. Especially, the conditions of Corollaries 1 and 2 are easily verified. Therefore, our results play an important role in the design of globally exponentially stable neural circuits and periodic oscillatory neural circuits.

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